Proofs, Exercises and Literature - Linear Algebra Recap

1 Exercises

1.1 Trivia Questions from the Lecture

1. $A \in \mathbb{R}^{n \times r}$, $B \in \mathbb{R}^{m \times r}$, which product is well-defined?
   - A. BA
   - B. $A^\top B$
   - C. $AB^\top$
2. $A \in \mathbb{R}^{n \times r}$, $B \in \mathbb{R}^{m \times r}$, what is equal to $(AB^\top)^\top$?
   - A. $A^\top B$
   - B. $B^\top A^\top$
   - C. $BA^\top$
3. What is the matrix product computed by $C_{ji} = \sum_{s=1}^{r} A_{is} B_{js}$?
   - A. $C = AB^\top$
   - B. $C = B^\top A$
   - C. $C = BA^\top$
4. $A, B \in \mathbb{R}^{n \times n}$ have an inverse A^{-1}, B^{-1}, what is generally **not** equal to $AA^{-1}B$?
   - A. $A^{-1}BA$
   - B. B
   - C. $BB^{-1}B$
5. Let $v, w \in \mathbb{R}^d$, $\alpha \in \mathbb{R}$, then $\|\alpha v + w\| \leq$
   - A. $\alpha\|v + w\|$
   - B. $|\alpha|\|v\| + \|w\|$
   - C. $\alpha\|v\| + \|w\|$
6. Let $A, B \in \mathbb{R}^{n \times r}$, $\alpha \in \mathbb{R}$, then $\|A\| \leq$
   - A. $\|A - B\| + \|B\|$
   - B. $\alpha\|\frac{1}{\alpha}A\|$
   - C. $\|A\|^2$
7. Let $A, B \in \mathbb{R}^{n \times n}$, A is orthogonal, what is **not** equal to $\mathrm{tr}(ABA^\top)$?
   - A. $\mathrm{tr}(A^\top BA)$
   - B. $\mathrm{tr}(B)$
   - C. $\mathrm{tr}(ABA)$

1.2 Additional Exercises

1. Compute the matrix product AB inner-product-wise and outer-product-wise

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

2. You have observations of 5 symptoms of a disease for three patients represented in the binary matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

   Compute the matrix $AA^\top$ and $A^\top A$ and interpret the result with regard to the scenario.

3. Find a matrix/vector notation to compute the vector of average feature values for a matrix $A \in \mathbb{R}^{n\times d}$, representing n observations of d features. Make an example for your computation.

4. Show that $\|A - B\|^2 = -2\,\mathrm{tr}(AB^\top) + 2n$ for orthogonal matrices $A, B \in \mathbb{R}^{n\times n}$.

5. Show that the following norms are orthogonal invariant
   - the vector L_2-norm
   - the Frobenius norm (matrix L_2-norm)
   - the operator norm

2 Recommended Literature

As always, the best exercise is to go through the lecture and see if you can follow the steps with pen and paper and to make the exercises. The linear algebra lecture is tailored to the needs for this lecture. If you want a more general and extensive overview, the following material is recommended.

Linear Algebra and Optimization for Machine Learning by Charu C. Aggarwal

Chapter one, in particular Sections 1.1-1.3 give a good introduction to vector spaces and matrices, norms and matrix multiplication. If you want to go deeper into the subject of linear algebra, then I would recommend to have a look at the Sections 2.1-2.4 and 7.1 and 7.2 as well.

The Course Linear Algebra and Applications (2DBI00) from Michiel Hochstenbach

Michiel is giving a very good course at TU/e about linear algebra and applications, where the applications are often data mining/machine learning problems. You can see the video lectures of 2018/2019 at the videocollege. Search for 2DBI00 in channels (not videos!). Select 2018-2019 to watch the latest recorded videos.